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Bernstein theorems for space-like graphs with parallel mean curvature and controlled growth☆

Yuxin Dong*

Institute of Mathematics, Fudan University, Shanghai 200433, PR China
Key Laboratory of Mathematics for Nonlinear Sciences, Ministry of Education, PR China

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Abstract

In this paper, we obtain an Ecker–Huisken-type result for entire space-like graphs with parallel mean curvature.

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1. Introduction

In 1914, Bernstein proved that the only entire minimal graph in R^3 is a plane. This result was generalized to R^{m+1} for $m \leq 7$, and higher dimensions and co-dimensions under various growth conditions, see [7,12,15,16] and their references. In 1965, Chern [3] showed that the only entire graphic hypersurface in R^{m+1} with constant mean curvature must be minimal. Therefore we have the corresponding Bernstein-type results for constant mean curvature hypersurfaces. Bernstein-type results for submanifolds in R^{m+n} with parallel mean curvature were also obtained by some authors (cf. [8,9] and [6]).

In 1968, Calabi [2] raised a similar problem for extremal hypersurfaces in Lorentz–Minkowski space R_1^{m+1} and he proved that the Bernstein result is true for $2 \leq m \leq 5$. Later, Cheng and Yau [5] extended Calabi's result to all m as follows: The only complete extremal space-like hypersurfaces in R_1^{m+1} are space-like hyperplanes. Recently, Jost and Xin [10] generalized this result to higher co-dimensional case.

On the other hand, it is important to investigate space-like constant mean curvature hypersurfaces in R_1^{m+1}, which have interest in relativity theory (cf. [11]). In [13], Treibergs showed that there are many entire space-like graphs with constant mean curvature besides hyperboloids. Thus a Chern-type result is no longer true in this case. It is known that the Gauss map of a constant mean curvature space-like hypersurface M is a harmonic map to hyperbolic space. Xin [17] got a Bernstein result by assuming the boundedness of the Gauss map. Later, [19] and [4] extended this result by proving that M must be a space-like hyperplane if its Gauss image lies in a horoball in the hyperbolic

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* Corresponding address: Institute of Mathematics, Fudan University, Shanghai 200433, PR China. Tel.: +86 21 6551 9798.
E-mail address: yxdong@fudan.edu.cn.

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space. Another natural generalization is to consider a space-like submanifold in pseudo-Euclidean space R_n^{m+n} with parallel mean curvature. In [18] the author extended the result in [17] mentioned previously to the case of higher co-dimensions under the same boundedness assumption on Gauss map.

In this paper, we consider a space-like graphic submanifold $M = \{(x, f(x)) : x \in R^m\}$ in R_n^{m+n} with parallel mean curvature. Since M is space-like, the induced metric $(g_{ij}) = (\delta_{ij} - \sum_{s=1}^n f_{x_i}^s f_{x_j}^s)$ is positive definite. Set

$$*\Omega = \left\{ \sqrt{\det\left(I - \sum_{s=1}^n f_{x_i}^s f_{x_j}^s \right)} \right\}^{-1}.$$

Our main result is the following:

Theorem. *Let $M^m = (x, f(x))$ be an entire space-like graph in R_n^{m+n} with parallel mean curvature. If the function $*\Omega$ has growth*

$$*\Omega = o(r) \quad \text{as } r \to \infty$$

where $r = \sqrt{\sum_{i=1}^m x_i^2}$, then M is a space-like m-plane.

Our strategy is to establish a Chern-type result for an entire space-like graph with parallel mean curvature under the growth condition of $*\Omega$. Then the result follows immediately from [5] and [10]. Notice that the Gauss image of M is bounded if and only if $*\Omega$ is bounded. On the other hand, if $n = 1$, we have

$$*\Omega = \frac{1}{\sqrt{1 - |\nabla f|^2}}.$$

Therefore the growth condition of $*\Omega$ is similar to that one given by Ecker–Husken [7] for minimal graphic hypersurfaces in R^{m+1}. The above result may also be regarded as an Ecker–Huisken-type result for space-like graphs with parallel mean curvature. By calculating the quantity $*\Omega$ of the hyperboloid, we will see that the growth condition is optimal. In [6], the author uses a similar method to establish some Bernstein-type results for submanifolds in Euclidean space with parallel mean curvature.

2. Preliminaries

In this section, we will generalize Chern's method [3] to our setting. Let R_n^{m+n} be an $(m+n)$-dimensional pseudo-Euclidean space of index n, namely the vector space R^{m+n} endowed with the metric

$$(\,,\,) = (\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_m)^2 - (\mathrm{d}x_{m+1})^2 - \cdots - (\mathrm{d}x_{m+n})^2. \tag{1}$$

The standard Euclidean metric of R^{m+n} will be denoted by $(\,,\,)_E$. For a vector v in R^{m+n}, we will use the notations $|v|$ and $|v|_E$ to denote the norms of v with respect to $(\,,\,)$ and $(\,,\,)_E$ respectively.

Let $z : M^m \to R_n^{m+n}$ be a space-like immersion of an oriented m-dimensional manifold into R_n^{m+n}. We will regard z as a vector-valued function on M. Choose a local Lorentzian frame field $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}\}$ such that $\{e_{m+1}, \ldots, e_{m+n}\}$ is a normal frame field of M. Throughout this paper, we agree with the following indices:

$$\begin{aligned} &1 \le A, B, C \ldots \le m+n \\ &1 \le i, j, k, \ldots \le m, \quad m+1 \le \alpha, \beta, \gamma, \ldots \le m+n. \end{aligned} \tag{2}$$

Write

$$\begin{aligned} \mathrm{d}z &= \sum_A \omega_A e_A \\ \mathrm{d}e_A &= \sum_B \omega_{AB} e_B. \end{aligned} \tag{3}$$

Therefore $\{\omega_i\}$ is a dual frame field of $\{e_i\}$ and $\omega_\alpha = 0$ on M. The induced Riemannian metric of M is then given by $\mathrm{d}s_M^2 = \sum_i \omega_i^2$. By Cartan's lemma, we have

$$\omega_{\alpha i} = \sum_k h_{\alpha ij}\omega_j, \qquad h_{\alpha ij} = h_{\alpha ji} \tag{4}$$

where $h_{\alpha ij}$ are components of the second fundamental form of M in R_n^{m+n}. The mean curvature vector of M is defined by

$$\vec{H} = \frac{1}{m}\sum_{\alpha,k} h_{\alpha kk} e_\alpha. \tag{5}$$

If $\nabla^\perp \vec{H} = 0$, M is said to have parallel mean curvature. If $\vec{H} = 0$, M is called an extremal space-like submanifold.

Now we consider a space-like graph $M = \{(x, f(x)) : x \in D \subset R^m\}$ in R_n^{m+n} with parallel mean curvature $\vec{H}$, where D is a compact domain with smooth boundary ∂D. Obviously $H = \sqrt{-(\vec{H}, \vec{H})}$ is a nonnegative constant.

Let $\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ be the parallel m-form on R_n^{m+n} and let $\{a_1, \ldots, a_{m+n}\}$ be an oriented Lorentzian basis of R_n^{m+n} such that $\{a_i\}_{i=1}^m$ is an oriented orthonormal basis of R^m. If $H > 0$, we have a global future-directed normal vector field $e_H = H^{-1}\vec{H}$. Therefore we may define a global m-form on M as follows:

$$\Phi = (m-1)!\sum_{i=1}^m \mathrm{d}(a_1, z) \wedge \cdots \wedge \mathrm{d}(a_{i-1}, z) \wedge \mathrm{d}(a_i, e_H) \wedge \mathrm{d}(a_{i+1}, z) \wedge \cdots \wedge \mathrm{d}(a_m, z). \tag{6}$$

Clearly Φ is independent of the choice of the oriented orthogonal basis $\{a_i\}_{i=1}^m$ in R^m.

For any $p \in M$ then the differential of f is a linear map from R^m to R^n. As in [14], we can use singular value decomposition to find orthonormal bases $\{a_i\}_{i=1}^m$ for R^m and $\{a_\alpha\}_{\alpha=m+1}^{m+n}$ for R^m such that

$$\mathrm{d}f(a_i) = \lambda_i a_{m+i} \tag{7}$$

for $i = 1, \ldots, m$. Notice that $\lambda_i = 0$ if $i > \min\{m, n\}$. Then we have

$$(a_i, a_j) = \delta_{ij}, \qquad (a_i, a_\alpha) = 0, \qquad (a_\alpha, a_\beta) = -\delta_{\alpha\beta}.$$

Therefore we have a Lorentzian basis $\{e_A\}$ at p given by

$$\left\{ e_i = \frac{1}{\sqrt{1-\lambda_i^2}}(a_i + \lambda_i a_{m+i}) \right\}_{i=1,\ldots,m} \in T_pM \tag{8}$$

and

$$\left\{ e_\alpha = \frac{1}{\sqrt{1-\lambda_{\alpha-m}^2}}(a_\alpha + \lambda_{\alpha-m} a_{\alpha-m}) \right\}_{\alpha=m+1}^{m+n} \in T_p^\perp M. \tag{9}$$

By definition $*\Omega = \Omega(e_1, \ldots, e_m)$, and thus we have

$$*\Omega = \frac{1}{\sqrt{\prod_{i=1}^m (1-\lambda_i^2)}}. \tag{10}$$

Lemma 1. *Under the above notations, we have*

$$\Phi = m!H(*\Omega)\omega^1 \wedge \cdots \wedge \omega^m$$

where $\omega^1 \wedge \cdots \wedge \omega^m$ *is volume form of* M.

Proof. Using (3), (8) and (9), we have from (6) the following:

$$\begin{aligned}\Phi &= (m-1)!\sum_{i=1}^{m}(a_1,\mathrm{d}z)\wedge\cdots\wedge(a_{i-1},\mathrm{d}z)\wedge(a_i,\mathrm{d}e_H)\wedge(a_{i+1},\mathrm{d}z)\wedge\cdots\wedge(a_m,\mathrm{d}z)\\
&= (m-1)!\sum_{i=1}^{m}\frac{1}{\sqrt{1-\lambda_1^2}}\omega^1\wedge\cdots\wedge\frac{1}{\sqrt{1-\lambda_{i-1}^2}}\omega^{i-1}\wedge\left(\frac{-h_{ii}^H}{\sqrt{1-\lambda_i^2}}\omega^i\right)\\
&\quad\wedge\frac{1}{\sqrt{1-\lambda_{i+1}^2}}\omega^{i+1}\wedge\cdots\wedge\frac{1}{\sqrt{1-\lambda_m^2}}\omega^m\\
&= -(m-1)!(*\Omega)\left(\sum_i h_{ii}^H\right)\omega^1\wedge\cdots\wedge\omega^m\\
&= m!(*\Omega)H\omega^1\wedge\cdots\wedge\omega^m\end{aligned}$$

where $\sum_i h_{ii}^H = \langle\sum_\alpha h_{\alpha ii}e_\alpha, e_H\rangle = -mH$. This proves the lemma. □

We may write

$$\Phi = (m-1)!\mathrm{d}\alpha \tag{11}$$

where

$$\alpha = \sum_i(-1)^{i-1}(a_i,e_H)\mathrm{d}(a_1,z)\wedge\cdots\wedge\mathrm{d}(a_{i-1},z)\wedge\mathrm{d}(a_{i+1},z)\wedge\cdots\wedge\mathrm{d}(a_m,z). \tag{12}$$

Applying the Stokes Theorem to (11), we get

$$mH\int_M(*\Omega)\omega^1\wedge\cdots\wedge\omega^m = \int_{\partial M}\alpha. \tag{13}$$

We project $z(M)$ orthogonally into the m-plane spanned by $\{a_i\}_{i=1}^m$. If $z'(p)$ is the image point of $z(p)$, $p\in M$, under this orthogonal projection, we have

$$z' = z + \sum_{\alpha=m+1}^{m+n}(a_\alpha,z)a_\alpha. \tag{14}$$

Let Ψ be a nonzero differential form on $\partial M = \{(x,f(x)) : x\in\partial D\}$, defined locally. Using this form, the elements of volume of $z'(\partial M)$, $z(\partial M)$ may be expressed respectively as $P\Psi$, $Q\Psi$ with $P\geq 0$ and $Q\geq 0$. Let $\alpha = R\Psi$. We write

$$\omega_{i_1}\wedge\cdots\wedge\omega_{i_{m-1}} = p_{i_1,\ldots,i_{m-1}}\Psi \tag{15}$$

on ∂M.

By a direct computation, we have

$$\begin{aligned}\frac{1}{(m-1)!}\underbrace{\mathrm{d}z\wedge\cdots\wedge\mathrm{d}z}_{m-1} &= \frac{1}{(m-1)!}\left(\sum\omega_{i_1}e_{i_1}\right)\wedge\cdots\wedge\left(\sum\omega_{i_{m-1}}e_{i_{m-1}}\right)\\
&= \sum_i p_{1,\ldots,i-1,i+1,\ldots,m}(e_1\wedge\cdots\wedge e_{i-1}\wedge e_{i+1}\wedge\cdots\wedge e_m)\Psi\end{aligned}$$

so that

$$Q^2 = \sum_i p^2_{1,\ldots,i-1,i+1,\ldots,m}. \tag{16}$$

Using (8)–(10), we get:

$$\begin{aligned}\alpha &= \sum_{i=1}^{m}(-1)^{i-1}(a_i, e_H)(a_1, \mathrm{d}z) \wedge \cdots \wedge (a_{i-1}, \mathrm{d}z) \wedge (a_{i+1}, \mathrm{d}z) \wedge \cdots \wedge (a_m, \mathrm{d}z)\\ &= \sum_{i=1}^{m}(-1)^{i-1}(a_i, e_H)\prod_{j\neq i}\frac{1}{\sqrt{1-\lambda_j^2}}\omega^1 \wedge \cdots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \cdots \wedge \omega^m\\ &= \sum_{i=1}^{m}(-1)^{i-1}(a_i, e_H)p_{1\ldots i-1,i+1,\ldots,m}\Psi\\ &= (*\Omega)\sum_{i=1}^{m}(-1)^{i}\lambda_i\xi_{m+i}p_{1\ldots i-1,i+1,\ldots,m}\Psi \end{aligned} \tag{17}$$

where $\xi_{m+k} = \langle e_H, e_{m+k}\rangle$ if $k \leq \min\{m, n\}$ and $\xi_{m+k} = 0$ if $k > \min\{m, n\}$. Obviously $\sum_{i=1}^{m}\xi_{m+i}^2 \leq 1$. Since $|\mathrm{d}f| < 1$, we get from (17) and the Cauchy–Schwarz inequality that

$$|R| \leq (*\Omega)Q. \tag{18}$$

Next, we will show that if M is a space-like hypersurface, there is a nice formula relating the quantities P, Q and R. When $n = 1$, (12) is simplified to

$$\alpha = vp_{2,\ldots,m} \tag{19}$$

where $v = \lambda_1/\sqrt{1-\lambda_1^2} = \lambda_1(*\Omega)$; and thus

$$R^2 = v^2p_{2,\ldots,m}^2. \tag{20}$$

Write $a = a_{m+1}$. Then (14) becomes

$$x' = x + (a, x)a. \tag{21}$$

From (9) and (10), we easily derive

$$(a, e_i) = -\delta_{i1}v, \qquad (a, e_{m+1}) = -*\Omega \tag{22}$$

and

$$a = -ve_1 + *\Omega e_{m+1}. \tag{23}$$

To determine P, we compute $\frac{1}{(m-1)!}\underbrace{\mathrm{d}z' \wedge \cdots \wedge \mathrm{d}z'}_{m-1}$ as follows:

$$\begin{aligned}&\frac{1}{(m-1)!}\underbrace{\mathrm{d}z' \wedge \cdots \wedge \mathrm{d}z'}_{m-1}\\ &\quad= \frac{1}{(m-1)!}\left\{\sum_{i_1}e_{i_1}\omega_{i_1} - v\omega_1 a\right\} \wedge \cdots \wedge \left\{\sum_{i_{m-1}}e_{i_{m-1}}\omega_{i_{m-1}} - v\omega_1 a\right\}\\ &\quad= \frac{1}{(m-1)!}\sum\omega_{i_1} \wedge \cdots \omega_{i_{m-1}}(e_{i_1} \wedge \cdots \wedge e_{i_{m-1}}) - \frac{v}{(m-1)!}\sum_{1\leq s\leq m-1}\omega_{i_1}\\ &\qquad \wedge \cdots \wedge \omega_{i_{s-1}} \wedge \omega_1 \wedge \omega_{i_{s+1}} \wedge \cdots \wedge \omega_{i_{m-1}}(e_{i_1} \wedge \cdots \wedge a \wedge \cdots \wedge e_{i_{m-1}})\\ &\quad= \frac{1}{(m-1)!}\sum\omega_{i_1} \wedge \cdots \omega_{i_{m-1}}(e_{i_1} \wedge \cdots \wedge e_{i_{m-1}}) + \frac{v^2}{(m-1)!}\sum_{1\leq s\leq m-1}(-1)^{s-1}\omega_1\\ &\qquad \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{s-1}} \wedge \omega_{i_{s+1}} \cdots \wedge \omega_{i_{m-1}}(e_{i_1} \wedge \cdots \wedge e_1 \wedge \cdots \wedge e_{i_{m-1}})\end{aligned}$$

$$-\frac{v*\Omega}{(m-1)!}\sum_{1\le s\le m-1}(-1)^{s-1}\omega_1\wedge\omega_{i_1}\wedge\cdots\wedge\omega_{i_{s-1}}\wedge\omega_{i_{s+1}}\cdots\wedge\omega_{i_{m-1}}(e_{i_1}\wedge\cdots\wedge e_{m+1}\wedge\cdots\wedge e_{i_{m-1}}).$$

So the coefficient of Ψ in $\frac{1}{(m-1)!}\underbrace{\mathrm{d}z'\wedge\cdots\wedge\mathrm{d}z'}_{m-1}$ is

$$\sum_{i_1<\cdots<i_{m-1}}p_{i_1\ldots i_{m-1}}e_{i_1}\wedge\cdots\wedge e_{i_{m-1}}+v^2\sum_{1<i_2<\cdots<i_{m-1}}p_{1i_2\ldots i_{m-1}}e_1\wedge e_{i_2}\wedge\cdots\wedge e_{i_{m-1}}$$
$$-(-1)^m v(*\Omega)\sum_{i_2<\cdots<i_{m-1}}p_{1i_2\ldots i_{m-1}}e_{i_2}\wedge\cdots\wedge e_{i_{m-1}}\wedge e_{m+1}.$$

It follows that

$$\begin{aligned}P^2&=\sum_{1<i_1<\cdots<i_{m-1}}p^2_{i_1\ldots i_{m-1}}+(1+v^2)^2\sum_{1<i_2<\cdots<i_{m-1}}p^2_{1i_2\ldots i_{m-1}}\\&\quad-v^2(*\Omega)^2\sum_{1<i_2<\cdots<i_{m-1}}p^2_{1i_2\ldots i_{m-1}}\\&=\sum_{i_1<\cdots<i_{m-1}}p^2_{i_1\ldots i_{m-1}}+v^2\sum_{1<i_2<\cdots<i_{m-1}}p^2_{1i_2\ldots i_{m-1}}\\&=\sum_{i_1<\cdots<i_{m-1}}p^2_{i_1\ldots i_{m-1}}+v^2\left(\sum_{i_1<\cdots<i_{m-1}}p^2_{i_1\ldots i_{m-1}}-p^2_{2\ldots m}\right)\\&=(1+v^2)Q^2-R^2\end{aligned}$$

since $(e_{m+1},e_{m+1})=-1$ and $v^2-(*\Omega)^2=-1$. Thus we have

$$P^2+R^2=(*\Omega)^2Q^2. \tag{24}$$

3. Bernstein-type theorems

In this section, we take $D=\left\{x\in R^m:\sum_{i=1}^m x_i^2\le r\right\}$. As before, let

$$M=\{(x,f(x)):x\in D\}$$

be a space-like graph in R^{m+n}_n with parallel mean curvature.

Lemma 2. *On ∂M, we have*

$$Q\le P. \tag{25}$$

In particular, if $n=1$, i.e., M is a space-like hypersurface, then we have

$$Q\le\sqrt{(*\Omega)^{-2}+|\mathrm{d}f(\eta_m)|^2}\,P \tag{26}$$

where $|\mathrm{d}f(\eta_m)|$ may be regarded as the radial singular value of the map f.

Proof. Choose an orthonormal basis $\{\eta_1,\ldots,\eta_m\}$ at $q\in\partial D$ in R^m such that η_m is a normal vector of ∂D. We have the corresponding tangent vectors of the graph M at $(q,f(a))$

$$\xi_i=(\eta_i,\mathrm{d}f(\eta_i)),\quad i=1,\ldots,m.$$

It is easy to see that

$$|\xi_1\wedge\cdots\wedge\xi_m|=(*\Omega)^{-1} \tag{27}$$

and

$$|\xi_1\wedge\cdots\wedge\xi_{m-1}|=Q/P. \tag{28}$$

Write

$$\xi_i = \widetilde{\eta_i} + \widetilde{\mathrm{d}f(\eta_i)}$$

where $\widetilde{\eta_i} = (\eta_i, 0)$ and $\widetilde{\mathrm{d}f(\eta_i)} = (0, \mathrm{d}f(\eta_i))$. Therefore

$$|\xi_i|^2 = |\widetilde{\eta_i}|_E^2 - |\widetilde{\mathrm{d}f(\eta_i)}|_E^2 = 1 - |\widetilde{\mathrm{d}f(\eta_i)}|_E^2 \leq 1$$

and thus

$$|\xi_1 \wedge \cdots \wedge \xi_{m-1}| \leq 1. \tag{29}$$

From (28) and (29), we have (25).

Now assume that $n = 1$. Obviously ξ_i, $i = 1, \ldots, m-1$, and $\widetilde{f(\eta_m)}$ are tangent to the cylinder

$$C_D = \left\{ (x_1, \ldots, x_{m+n}) \in R^{m+n} : \sum_{i=1}^{m} x_i^2 = r \right\}$$

and the horizontal vector $\widetilde{\eta_m}$ is orthogonal to C_D at the point $(q, f(q))$. It follows that

$$\begin{aligned}
(*\Omega)^{-2} &= |\xi_1 \wedge \cdots \wedge \xi_m|^2 \\
&= |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \widetilde{\eta_m} + \xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \widetilde{\mathrm{d}f(\eta_m)}|^2 \\
&= |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \widetilde{\eta_m}|^2 + |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \widetilde{\mathrm{d}f(\eta_m)}|^2 \\
&= |\xi_1 \wedge \cdots \wedge \xi_{m-1} \wedge \widetilde{\eta_m}|^2 + |\widetilde{\eta_1} \wedge \cdots \wedge \widetilde{\eta_{m-1}} \wedge \widetilde{\mathrm{d}f(\eta_m)}|^2 \\
&\geq |\xi_1 \wedge \cdots \wedge \xi_{m-1}|^2 - |\widetilde{\mathrm{d}f(\eta_m)}|_E^2
\end{aligned}$$

i.e.,

$$Q^2/P^2 = |\xi_1 \wedge \cdots \wedge \xi_{m-1}|^2 \leq (*\Omega)^{-2} + |\widetilde{df(\eta_m)}|_E^2.$$

This gives (26). □

We recall the following

Theorem A ([5,10]). *Let M be a space-like m-submanifold in R_n^{m+n} with parallel mean curvature. Assume that M is closed with respect to the Euclidean topology. Then M is complete with respect to the induced metric from the ambient space. In particular, if M is a complete extremal space-like m-submanifold in R_n^{m+n}, then M has to be a space-like m-plane.*

Remark. Obviously, an entire graph is closed with respect to the Euclidean topology. Hence we know that any entire extremal space-like graph must be a space-like m-plane.

Theorem 1. *Let $M^m = (x, f(x))$ be an entire space-like graph in R_n^{m+n} with parallel mean curvature. If $*\Omega$ has the following growth*

$$*\Omega = o(r) \quad \text{as } r \to \infty \tag{30}$$

where $r = \sqrt{\sum_{i=1}^m x_i^2}$, then M is a space-like m-plane.

Proof. Let $M_r = \{(x, f(x)) : x \in D_r \subseteq R^m\}$, where D_r denotes the closed ball of radius r centered at the origin in R^m. From (13), (18) and Lemma 2, we have

$$\begin{aligned}
mH \int_{M_r} (*\Omega)\omega^1 \wedge \cdots \wedge \omega^m &\leq \int_{\partial M_r} R\Psi \\
&\leq \int_{\partial M_r} *\Omega P \Psi \\
&\leq \sup_{\partial D_r}\{*\Omega\} \mathrm{Vol}(\partial D_r)
\end{aligned}$$

i.e.,

$$mH\mathrm{Vol}(D_r) \le \sup_{\partial D_r}\{*\Omega\}\mathrm{Vol}(\partial D_r).$$

Thus

$$H \le C\frac{\sup_{\partial D_r}\{*\Omega\}}{r}$$

where C is a universal constant. Let $r \to \infty$. It follows that $H \equiv 0$. Hence we may complete the proof by Theorem A. □

For space-like hypersurfaces, we may give a more delicate growth condition to ensure the above result.

Proposition 2. *Let $M^m = (x, f(x))$ be an entire space-like hypersurface in R_1^{m+1} with constant mean curvature. If*

$$\sup_{\partial D_r}\{|\mathrm{d}f(\eta_m)|_E * \Omega\} = o(r)$$

where $r = \sqrt{\sum_{i=1}^m x_i^2}$, then M is a space-like m-plane.

Proof. From (13), (24), (25) and Lemma 2, we have

$$\begin{aligned} mH\int_{M_r}(*\Omega)\omega^1\wedge\cdots\wedge\omega^m &\le \int_{\partial M_r} R\Psi \\ &\le \int_{\partial M_r}\sqrt{(*\Omega)^2Q^2 - P^2}\,\Psi \\ &\le \int_{\partial M_r}|\mathrm{d}f(\eta_m)|_E(*\Omega)P\,\Psi \\ &\le \sup_{\partial D_r}\{|\mathrm{d}f(\eta_m)|_E * \Omega\}\mathrm{Vol}(\partial D_r). \end{aligned}$$

By the same argument as in Theorem 1, we prove the proposition. □

Let us consider a typical example of space-like graphs in R_1^{m+1} with constant mean curvature.

Example 1. The hyperboloid is defined by

$$\begin{aligned} H_{-1}^m &= \left\{(x_1,\ldots,x_n,x_{n+1}) \in R_1^{m+1} : \sum_{i=1}^m x_i^2 - x_{m+1}^2 = -1, x_{m+1} \ge 0\right\} \\ &= \left\{(x, f(x)) : f = \sqrt{1+\sum_{i=1}^m x_i^2}, x \in R^m\right\}. \end{aligned}$$

By a direct computation, we have

$$*\Omega = \frac{1}{\sqrt{1-|\nabla f|^2}} = \sqrt{1+\sum_{i=1}^m x_i^2} = O(r). \tag{31}$$

From (31), we see that the growth condition in Theorem 1 is optimal.

Theorem B (*[19,4]*). *Let $M^m = (x, f(x))$ be a complete space-like hypersurface in R_1^{m+1} with constant mean curvature. If the image of the Gauss map $\gamma : M \to H^m(-1)$ lies in a horoball in $H^m(-1)$, then M must be a space-like hyperplane.*

It is known that every complete space-like hypersurface in R_1^{m+1} is spatially entire (cf. [1]). To compare Theorem 1 with Theorem B, we hope to find the equivalent restriction on the function $*\Omega$, if the image of γ lies in a horoball.

Let $M = (x, f(x))$ be a space-like graphic hypersurface in R_1^{m+1}. Its Gauss map γ is given by

$$\begin{aligned} \gamma : M &\longrightarrow H_{-1}^m \\ x &\longmapsto \frac{1}{\sqrt{1-|\nabla f|^2}}(f_{x_1}, \ldots, f_{x_m}, 1) = *\Omega(f_{x_1}, \ldots, f_{x_m}, 1) \end{aligned} \tag{32}$$

where H_{-1}^m is the hyperboloid endowed with the induced metric from R_1^{m+1}. Obviously the Gauss image of M is bounded in H_{-1}^m if and only if $*\Omega$ is bounded. This also holds true for higher co-dimensional case (cf. [18]).

It is easier to use the upper half-space model H^m of the hyperbolic space for describing horoballs. We consider the following maps

$$\begin{aligned} h_1 : H_{-1}^m &\longrightarrow B^m \\ (x_1, \ldots, x_m, x_{m+1}) &\longmapsto \left(\frac{x_1}{1+x_{m+1}}, \ldots, \frac{x_m}{1+x_{m+1}}\right) \end{aligned} \tag{33}$$

and

$$\begin{aligned} h_2 : B^m &\longrightarrow H^m = \left\{(y_1, \ldots, y_m) \in R^m : y_m > 0\right\} \\ p &\longmapsto 2\frac{p-p_0}{|p-p_0|^2} - (0, \ldots, 0, 1) \end{aligned} \tag{34}$$

where $p_0 = (0, \ldots, -1)$ and H^m is endowed with the metric $g = y_m^{-2}(dy_1^2 + \cdots + dy_m^2)$. The set $\{(y_1, \ldots, y_m) \in H^m : y_m > c > 0\}$ for any positive constant c is a horoball in H^m. It is known that $h_2 \circ h_1 : H_{-1}^m \to H^m$ is an isomorphism. From (32)–(34), we may get the mth component of $h_2 \circ h_1 \circ \gamma$ as follows:

$$(h_2 \circ h_1 \circ \gamma)_m = \frac{1}{(1+f_{x_m}) * \Omega}.$$

So the condition $y_m > c > 0$ is equivalent to

$$(1+f_{x_m}) * \Omega < \frac{1}{c}. \tag{35}$$

Note that f_{x_m} may be replaced by any f_{x_i} or $v(f)$ which denotes the derivative in any fixed unit direction v in R^m. Obviously, if there exists a sequence of points $\{p_k\}$ such that $*\Omega(p_k) \to \infty$, then $(f_{x_m})(p_k) \to -1$. Therefore (35) implies that all 'bad singular directions' approach one direction, i.e., $\partial/\partial x_m$.

Since $*\Omega = (\sqrt{1-|\nabla f|^2})^{-1}$, we see that the growth condition in Theorem 1 is very much like that one given by Ecker–Huisken in [7] for a minimal graphic hypersurface in the Euclidean space R^{m+1}. Hence Theorem 1 may be regarded as an Ecker–Huisken-type result.

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